THE RIESZ TRANSFORM FOR THE HARMONIC OSCILLATOR IN SPHERICAL COORDINATES

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Abstract. In this paper we show weighted estimates in mixed norm spaces for the Riesz transform associated with the harmonic oscillator in spherical coordinates. In order to prove the result we need a weighted inequality for a vector-valued extension of the Riesz transform related to the Laguerre expansions which is of independent interest. The main tools to obtain such extension are a weighted inequality for the Riesz transform independent of the order of the involved Laguerre functions and an appropriate adaptation of Rubio de Francia's extrapolation theorem.

1. Introduction

Let $H := -\Delta + |\cdot|^2$ be the harmonic oscillator in $\mathbb{R}^n$. The eigenfunctions of this operator in $\mathbb{R}^n$ verify $H\phi = E\phi$, where E is the corresponding eigenvalue. There are two complete sets of eigenfunctions for H. Using cartesian coordinates, one obtains the functions

$$\phi_k(x) = \prod_{i=1}^{n} h_{k_i}(x_i), \qquad k = (k_1, \dots, k_n) \in \mathbb{N}^n,$$

where $h_{k_i}(x_i) = (\sqrt{\pi}2^{k_i}k_i!)^{-1/2} H_{k_i}(x_i) e^{-x_i^2/2}$, and H_j denote the Hermite polynomials of degree $j \in \mathbb{N}$ (see [11, p. 60]). In this case $E_k = 2|k| + n$. The system of functions $\{\phi_k\}_{k\in\mathbb{N}^n}$ is orthonormal and complete in $L^2(\mathbb{R}^n, dx)$.

But the situation is completely different if we analyze the eigenfunctions of the harmonic oscillator by using spherical coordinates (see (2.1) below). Let $\mathbb{B}^n$ be the unit ball in $\mathbb{R}^n$ and $\mathbb{S}^{n-1} = \partial\mathbb{B}^n$. Let $\mathcal{H}_j$ be the space of spherical harmonics of degree j in n variables, and $\{\mathcal{Y}_{j,\ell}\}_{\ell=1,\dots,\dim\mathcal{H}_j}$ be an orthonormal basis for $\mathcal{H}_j$ in $L^2(\mathbb{S}^{n-1}, d\sigma)$, where σ is the surface area measure in $\mathbb{S}^{n-1}$. Then the eigenfunctions of the harmonic oscillator, see [7], are given by

(1.1)
$$\tilde{\phi}_{m,j,\ell}(x) = \left(\frac{2\Gamma(j+1)}{\Gamma(m-j+n/2)}\right)^{1/2} L_j^{n/2-1+m-2j}(r^2)\mathcal{Y}_{m-2j,\ell}(x)e^{-r^2/2}, \quad r = |x|,$$

where $m \geq 0$, $j = 0, \dots, [m/2]$, $\ell = 1, \dots, \dim\mathcal{H}_{m-2j}$, and L_j^b are Laguerre polynomials of order b and degree $j \in \mathbb{N}$, see [11, p. 76]. This system is orthonormal and

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complete in $L^2(\mathbb{R}^n, dx)$ and the eigenvalues are $E_{m,j,\ell} = (n+2m)$. Moreover

$$L^2(\mathbb{R}^n, dx) = \bigoplus_{m=0}^{\infty} \mathcal{J}_m$$

with

$$\mathcal{J}_m = \{f \in C^\infty(\mathbb{R}^n) : Hf = (n+2m)f\}.$$

The main target of this paper will be the analysis of the Riesz transform related to the system of eigenfunctions of the harmonic oscillator in spherical coordinates.

It could be said that the investigation of conjugacy operators related to discrete and continuous non-trigonometric orthogonal expansions was initiated in the seminal article by B. Muckenhoupt and E. M. Stein [13]. They analyze a substitute of classical conjugacy function in the context of ultraspherical polynomials expansions, Hankel transforms and Fourier-Bessel expansions. Later, the book by Stein [21] propelled the research in Fourier Analysis of general laplacians. It is noteworthy to observe that in the classical one-dimensional case the "continuous" counterpart of the conjugacy is the Hilbert transform, and we have the equivalence between Hilbert transform and the so called Riesz transform. So, abusing of the language, the wording conjugacy and Riesz transform are used as the same thing. For the last forty years, the research developed is huge, and the list of references could be endless. Concerning examples close to our context, the study of Riesz transforms in the setting of multi-dimensional Hermite functions was initiated by S. Thangavelu [23, 24] and continued in [22, 10, 12]. On the other hand, Riesz transforms associated with expansions based on different multi-dimensional Laguerre functions have been investigated by A. Nowak and K. Stempak in [15, 16].

We will analyze the Riesz transform associated with the system given in (1.1) in the so called mixed norm spaces $L^{p,2}(\mathbb{R}^n, r^{n-1}\,dr\,d\sigma)$ (see Section 2 for definition). These spaces arise frequently in harmonic analysis when the spherical harmonics are involved. The papers [20, 6, 4, 2] contain good examples of the use of these mixed norm spaces. In [5], the authors considered this kind of spaces to study the fractional integrals related to the functions $\tilde{\phi}_{m,j,\ell}$.

The boundedness properties of the Riesz transform related to $\tilde{\phi}_{m,j,\ell}$ in the mixed norm spaces will be reduced to two inequalities due to the decomposition of the harmonic oscillator in spherical coordinates. The first one will be a vector-valued inequality for a sequence of Riesz transforms for Laguerre expansions of convolution type and order $(n-2)/2+j$. This is the main point in the proof of our result and it requires very precise estimates of the kernel of the Riesz transform in the Laguerre setting in terms of the order. With these estimates we will be able to apply the Calderón-Zygmund theory. Then a suitable version of the extrapolation theorem of Rubio de Francia will do the rest to deduce the vector-valued extension. The second inequality appearing from the angular part of the harmonic oscillator, will be deduced from the Calderón-Zygmund theory as well.

2. Main result

The harmonic oscillator in spherical coordinates can be written as

$$H = -\frac{\partial^2}{\partial r^2} - \frac{n-1}{r}\frac{\partial}{\partial r} + r^2 - \frac{1}{r^2}\Delta_0, \tag{2.1}$$

where Δ_0 is the spherical part of the Laplacian. It can be checked that

$$H = \delta^*\delta + n,$$

with

$$\delta = \xi\left(\frac{\partial}{\partial r} + r\right) + \frac{1}{r}\nabla_0, \tag{2.2}$$

where ∇_0 is the angular part of the gradient, and

$$\delta^* = -\left(\frac{\partial}{\partial r} - r\right)\xi - \frac{1}{r}\operatorname{div}_0,$$

where div_0 is the angular part of the divergence.

For each $\sigma > 0$, we define the fractional integrals for the harmonic oscillator as

$$H^{-\sigma} f = \sum_{m=0}^{\infty} \frac{1}{(n+2m)^\sigma} \operatorname{Proj}_{\mathcal{J}_m} f,$$

where

$$\operatorname{Proj}_{\mathcal{J}_m} f = \sum_{j=0}^{\left[\frac{m}{2}\right]} \sum_{\ell=1}^{\dim \mathcal{H}_{m-2j}} c_{m,j,\ell}(f)\tilde{\phi}_{m,j,\ell}, \qquad c_{m,j,\ell}(f) = \int_{\mathbb{R}^n} \overline{\tilde{\phi}_{m,j,\ell}}(y) f(y)\, dy.$$

With the previous definitions the Riesz transform is given as

$$Rf = |\delta H^{-1/2} f|.$$

As we commented in the introduction, the appropriate spaces in order to analyze this kind of operators are the mixed norm spaces, defined as

$$L^{p,2}(\mathbb{R}^n, r^{n-1}\, dr\, d\sigma) = \{f(x) : \|f\|_{L^{p,2}(\mathbb{R}^n, r^{n-1}\, dr\, d\sigma)} < \infty\},$$

where

$$\|f\|_{L^{p,2}(\mathbb{R}^n, r^{n-1}\, dr\, d\sigma)} = \left(\int_0^\infty \left(\int_{\mathbb{S}^{n-1}} |f(rx')|^2\, d\sigma(x')\right)^{p/2} r^{n-1}\, dr\right)^{1/p},$$

with the obvious modification in the case $p = \infty$. The main feature of these spaces is that we consider the L^2-norm in the angular part and the L^p-norm in the radial. They are very different from $L^p(\mathbb{R}^n, dx)$; in fact $L^p(\mathbb{R}^n, dx) \subset L^{p,2}(\mathbb{R}^n, r^{n-1}\, dr\, d\sigma)$ for $p > 2$, $L^2(\mathbb{R}^n, dx) = L^{2,2}(\mathbb{R}^n, r^{n-1}\, dr\, d\sigma)$, and $L^{p,2}(\mathbb{R}^n, r^{n-1}\, dr\, d\sigma) \subset L^p(\mathbb{R}^n, dx)$ for $p < 2$. These spaces are the most suitable when spherical harmonics are involved due to the orthogonality of the system in the sphere $\mathbb{S}^{n-1}$. Indeed, if a function f on $\mathbb{R}^n$ is expanded in spherical harmonics,

$$f(x) = \sum_{j=0}^{\infty} \sum_{\ell=1}^{\dim \mathcal{H}_j} f_{j,\ell}(r)\mathcal{Y}_{j,\ell}\left(\frac{x}{r}\right),$$

where

$$f_{j,\ell}(r) = \int_{\mathbb{S}^{d-1}} f(rx')\overline{\mathcal{Y}_{j,\ell}}(x')\, d\sigma(x'),$$

we have

$$\|f\|_{L^{p,2}(\mathbb{R}^n, r^{n-1}\, dr\, d\sigma)} = \left\| \left(\sum_{j=0}^{\infty} \sum_{\ell=1}^{\dim \mathcal{H}_j} |f_{j,\ell}(r)|^2\right)^{1/2} \right\|_{L^p(\mathbb{R}_+, r^{n-1}\, dr)}.$$

To establish our result result related to weighted inequalities for the Riesz transform, we have to define the class of weights involved in them. For $1 \le p < \infty$, we

denote by $A_p^\alpha = A_p^\alpha(\mathbb{R}_+, d\mu_\alpha)$ the Muckenhoupt class of A_p weights on the space $(\mathbb{R}_+, d\mu_\alpha, |\cdot|)$, where

$$d\mu_\alpha(x) = x^{2\alpha+1}\,dx.$$

More precisely, A_p^α is the class of all nonnegative functions $w \in L^1_{\mathrm{loc}}(\mathbb{R}_+, d\mu_\alpha)$ such that $w^{-p'/p} \in L^1_{\mathrm{loc}}(\mathbb{R}_+, d\mu_\alpha)$, where $1/p + 1/p' = 1$, and

$$\sup_{I\in\mathcal{I}} \Big(\frac{1}{\mu_\alpha(I)}\int_I w\,d\mu_\alpha\Big)\Big(\frac{1}{\mu_\alpha(I)}\int_I w^{-p'/p}\,d\mu_\alpha\Big)^{p/p'} < \infty$$

when $1 < p < \infty$, or

$$\sup_{I\in\mathcal{I}} \frac{1}{\mu_\alpha(I)}\int_I w\,d\mu_\alpha \operatorname*{ess\,sup}_{x\in I} w^{-1} < \infty$$

if $p = 1$; here $\mathcal{I}$ is the class of all intervals in $(\mathbb{R}_+, |\cdot|)$.

The main result of the paper is stated below.

Theorem 2.1. *Let $n \ge 2$, $1 < p < \infty$, and $w \in A_p^{n/2-1}$. Then*

$$\|Rf\|_{L^{p,2}(\mathbb{R}^n, w(r)r^{n-1}\,dr\,d\sigma)} \le C\|f\|_{L^{p,2}(\mathbb{R}^n, w(r)r^{n-1}\,dr\,d\sigma)},$$

for each $f \in L^{p,2}(\mathbb{R}^n, w(r)r^{n-1}\,dr\,d\sigma)$ and with a constant C depending on n and w only.

The proof of Theorem 2.1 will be given in Section 5. The main estimates will be developed in Section 3 and Section 4.

3. Vector-valued inequalities for the Riesz transform for Laguerre expansions of convolution type

Let $\alpha > -1$, consider the differential operator given by

$$L_\alpha = -\frac{d^2}{dx^2} + x^2 - \frac{2\alpha+1}{x}\frac{d}{dx}, \tag{3.1}$$

which is symmetric on $\mathbb{R}_+$ equipped with the measure $d\mu_\alpha$ The Laguerre functions ℓ_k^α are defined by

$$\ell_k^\alpha(x) = \Big(\frac{2\Gamma(k+1)}{\Gamma(k+\alpha+1)}\Big)^{1/2} L_k^\alpha(x^2)e^{-x^2/2}, \quad x > 0,$$

where L_k^α are the Laguerre polynomials of degree $k \in \mathbb{N}$ and order $\alpha > -1$. The functions ℓ_k^α are eigenfunctions of the differential operator (3.1). Indeed, we have $L_\alpha \ell_k^\alpha = (4k + 2\alpha + 2)\ell_k^\alpha$. Furthermore, the system $\{\ell_k^\alpha\}_{k\in\mathbb{N}}$ is an orthonormal system in $L^2(\mathbb{R}_+, d\mu_\alpha)$. We will refer to the functions ℓ_k^α as *Laguerre functions of convolution type*.

It is easily seen that L_α can be decomposed as

$$L_\alpha = 2(\alpha+1) + \delta_\alpha^*\delta_\alpha,$$

where

$$\delta_\alpha = \frac{d}{dx} + x,$$

and

$$\delta_\alpha^* = -\frac{d}{dx} + x - \frac{2\alpha+1}{x}.$$

We provide now the definition of the Riesz transforms. Since the spectrum of L_α is separated from zero, for each $\sigma > 0$, we can define the fractional integrals of order σ by

$$(L_\alpha)^{-\sigma} f = \sum_{k=0}^{\infty} \frac{1}{(4k+2\alpha+2)^{\sigma}} \mathcal{P}_k^\alpha f, \tag{3.2}$$

with $\mathcal{P}_k^\alpha f = \langle f, \ell_k^\alpha \rangle_{d\mu_\alpha} \ell_k^\alpha$, where $\langle f, g\rangle_{d\mu_\alpha}$ means $\int_{\mathbb{R}_+} f(x)\overline{g(x)}\, d\mu_\alpha(x)$. Now, using $\frac{d}{dx} L_k^\alpha = -L_{k-1}^{\alpha+1}$, $\alpha > -1$, $k \in \mathbb{N}$, see [11, (4.18.6)], we obtain

$$\delta \ell_k^\alpha = -2\sqrt{k} x \ell_{k-1}^{\alpha+1}.$$

Therefore, for $f \in L^2(\mathbb{R}_+, d\mu_\alpha)$ with the expansion $f = \sum_k \langle f, \ell_k^\alpha\rangle_{d\mu_\alpha} \ell_k^\alpha$, we define the Riesz transform for the expansions of Laguerre functions of convolution type as

$$\mathcal{R}^\alpha f = \delta_\alpha (L_\alpha)^{-1/2} f = -2 \sum_{k=0}^{\infty} \Big(\frac{k}{4k+2\alpha+2} \Big)^{1/2} \langle f, \ell_k^\alpha \rangle_{d\mu_\alpha} x\, \ell_{k-1}^{\alpha+1}. \tag{3.3}$$

The system $\{x\, \ell_{k-1}^{\alpha+1}\}_{k\in\mathbb{N}}$ is an orthonormal basis in $L^2(\mathbb{R}_+, d\mu_\alpha)$, see [16, Proposition 4.1]. Therefore, the series above converges in $L^2(\mathbb{R}_+, d\mu_\alpha)$ and defines a bounded operator therein.

Our result about the Riesz transform for the Laguerre expansions is the following.

Theorem 3.1. *Let $\alpha \geq -1/2$, $a \geq 1$, and $1 < p, r < \infty$. Define $u_j(x) = x^{aj}$, $x \in \mathbb{R}_+$, $j = 0, 1, \ldots$. Then there exists a constant C such that*

$$\Big\| \Big(\sum_{j=0}^{\infty} |u_j \mathcal{R}^{\alpha+aj}(u_j^{-1} f_j)|^r \Big)^{1/r} \Big\|_{L^p(\mathbb{R}_+, w\, d\mu_\alpha)} \leq C \Big\| \Big(\sum_{j=0}^{\infty} |f_j|^r \Big)^{1/r} \Big\|_{L^p(\mathbb{R}_+, w\, d\mu_\alpha)},$$

for all $w \in A_p^\alpha$. Moreover the constant C depends on α and w only.

In order to prove Theorem 3.1, we need two ingredients.

Proposition 3.2. *Let $\alpha \geq -1/2$, $a \geq 1$, and $1 < p < \infty$. Define $u_j(x) = x^{aj}$, $x \in \mathbb{R}_+$, $j = 0, 1, \ldots$. Then,*

$$\int_0^\infty |u_j \mathcal{R}^{\alpha+aj}(u_j^{-1} f)(x)|^p w(x)\, d\mu_\alpha(x) \leq C \int_0^\infty |f(x)|^p w(x) d\mu_\alpha(x),$$

for every weight $w \in A_p^\alpha$ and with C independent of j and depending on α and w.

Proposition 3.3. *Let $\{T_j\}$ be a sequence of operators and suppose that, for some fixed $p > 1$, these operators are uniformly bounded in $L^p(\mathbb{R}_+, w\, d\mu_\alpha)$ for every weight $w \in A_p^\alpha$, i. e.*

$$\int_0^\infty |T_j f(x)|^p w(x)\, d\mu_\alpha(x) \leq C \int_0^\infty |f(x)|^p w(x)\, d\mu_\alpha(x), \tag{3.4}$$

with C independent of j. Then, the vector valued inequality

$$\Big\| \Big(\sum_{j=0}^{\infty} |T_j f_j|^r \Big)^{1/r} \Big\|_{L^p(\mathbb{R}_+, w\, d\mu_\alpha)} \leq C \Big\| \Big(\sum_{j=0}^{\infty} |f_j|^r \Big)^{1/r} \Big\|_{L^p(\mathbb{R}_+, w\, d\mu_\alpha)}, \tag{3.5}$$

holds for all $1 < r, p < \infty$ and $w \in A_p^\alpha$.

Therefore, the proof of Theorem 3.1 can be deduced easily by combining Proposition 3.2 and Proposition 3.3.

In Subsection 3.1 we will prove Proposition 3.2 and the proof of Proposition 3.3 will be given in Subsection 3.2.

Notation. The constants that do not depend on relevant quantities will be denoted by C and can change from one line to another without further comment. We also note that a constant denoted by C_α depends on α but not on j.

3.1. **Proof of Proposition 3.2.** The proof of Proposition 3.2 is based on the theory of Calderón-Zygmund operators defined on spaces of homogeneous type. The classical weighted Calderón-Zygmund theory is still valid, properly adjusted, in the setting of spaces of homogeneous type. The origin of the ideas can be found in the fundamental paper by Calderón [3]. We will write the operator $u_j\mathcal{R}^{\alpha+aj}(u_j^{-1}f)$ as an integral operator with kernel as follows

$$u_j\mathcal{R}^{\alpha+aj}(u_j^{-1}f)(x) = \int_0^\infty (xy)^{aj} R^{\alpha+aj}(x,y)f(y)\,d\mu_\alpha(y).$$

Then, we will prove growth and smoothness estimates for the kernel. These estimates are contained in the proposition below, that is the heart of the matter.

Proposition 3.4. *Let $\alpha \geq -1/2$, $a \geq 1$, and $j \geq 0$. Then*

$$|(xy)^{aj}R^{\alpha+aj}(x,y)| \leq \frac{C_1}{\mu_\alpha(B(x,|x-y|))}, \quad x \neq y, \tag{3.6}$$

$$|\nabla_{x,y}[(xy)^{aj}R^{\alpha+aj}(x,y)]| \leq \frac{C_2}{|x-y|\mu_\alpha(B(x,|x-y|))}, \quad x \neq y, \tag{3.7}$$

with C_1 and C_2 independent of j, and where $\mu_\alpha(B(x,|x-y|)) = \int_{B(x,|x-y|)} d\mu_\alpha$ and $B(x,|x-y|)$ is the ball of center x and radius $|x-y|$.

The case $j = 0$ of the previous proposition is contained in [16], so we will focus in the proof of (3.6) and (3.7) for $j \geq 1$ only.

The heat semigroup related to L_α is initially defined in $L^2(\mathbb{R}_+, d\mu_\alpha)$ as

$$T_{\alpha,t}f = \sum_{k=0}^\infty e^{-t(4k+2\alpha+2)}\langle f, \ell_k^\alpha\rangle_{d\mu_\alpha}\ell_k^\alpha, \quad t > 0.$$

We can write the heat semigroup $\{T_{\alpha,t}\}_{t>0}$ as an integral operator

$$T_{\alpha,t}f(x) = \int_0^\infty G_{\alpha,t}(x,y)f(y)\,d\mu_\alpha(y).$$

The Laguerre heat kernel is given by

$$G_{\alpha,t}(x,y) = \sum_{k=0}^\infty e^{-t(4k+2\alpha+2)}\ell_k^\alpha(x)\ell_k^\alpha(y).$$

The explicit expression for Laguerre heat kernel is known and it can be found in [11, (4.17.6)]:

$$G_{\alpha,t}(x,y) = (\sinh 2t)^{-1}\exp\Big(-\frac{1}{2}\coth(2t)(x^2+y^2)\Big)(xy)^{-\alpha}I_\alpha\Big(\frac{xy}{\sinh 2t}\Big),$$

with I_ν denoting the modified Bessel function of the first kind and order ν, see [11, Chapter 5].

It can be seen in [16, Section 3] that the Riesz transform (3.3) can be written as an integral operator

$$\mathcal{R}^\alpha f(x) = \int_0^\infty R^\alpha(x,y) f(y)\, d\mu_\alpha(y),$$

with the kernel

$$R^\alpha(x,y) = \frac{1}{\sqrt{\pi}} \int_0^\infty \delta_\alpha G_{\alpha,t}(x,y) t^{-1/2}\, dt.$$

Let us see now that the operators $u_j \mathcal{R}^{\alpha+aj}(u_j^{-1} f)$ are associated, in the Calderón-Zygmund sense, with the kernel given by

$$(xy)^{aj} R^{\alpha+aj}(x,y) = \frac{(xy)^{aj}}{\sqrt{\pi}} \int_0^\infty \delta_\alpha G_{\alpha+aj,t}(x,y) t^{-1/2}\, dt. \tag{3.8}$$

Proposition 3.5. *Let $\alpha \geq -1/2$, $a \geq 1$, and $u_j(x) = x^{aj}$, $x \in \mathbb{R}_+$, $j = 0, 1, \ldots$. Take $f, g \in C_c^\infty(\mathbb{R}_+)$ having disjoint supports. Let $u_j \mathcal{R}^{\alpha+aj}(u_j^{-1} f)$ be defined by* (3.3)*. Then*

$$\langle u_j \mathcal{R}^{\alpha+aj}(u_j^{-1} f), g\rangle_{d\mu_\alpha} = \int_0^\infty \int_0^\infty (xy)^{aj} R^{\alpha+aj}(x,y) f(y) \overline{g(x)}\, d\mu_\alpha(y)\, d\mu_\alpha(x).$$

Proof. Let $u_j^{-1} f = h_1$ and $u_j^{-1} g = h_2$. Then,

$$\begin{aligned}
\langle u_j \mathcal{R}^{\alpha+aj}(u_j^{-1} f), g\rangle_{d\mu_\alpha} &= \langle \mathcal{R}^{\alpha+aj} h_1, h_2\rangle_{d\mu_{\alpha+aj}} \\
&= \int_0^\infty \int_0^\infty R^{\alpha+aj}(x,y) h_1(y) \overline{h_2(x)}\, d\mu_{\alpha+aj}(y)\, d\mu_{\alpha+aj}(x) \\
&= \int_0^\infty \int_0^\infty (xy)^{aj} R^{\alpha+aj}(x,y) f(y) \overline{g(x)}\, d\mu_\alpha(y)\, d\mu_\alpha(x),
\end{aligned}$$

since the second identity above was proven in [16, Proposition 3.3]. □

We will find a suitable expression for the kernel (3.8), and this task boils down to expressing the corresponding heat kernel in an appropriate way. We use Schläfli's integral representation of Poisson's type for modified Bessel function, see [11, (5.10.22)],

$$I_\nu(z) = z^\nu \int_{-1}^1 \exp(-zs)\, \Pi_\nu(ds), \quad |\arg z| < \pi, \ \ \nu > -\frac{1}{2},$$

where the measure $\Pi_\nu(du)$ is given by

$$\Pi_\nu(du) = \frac{(1-u^2)^{\nu-1/2}\, du}{\sqrt{\pi} 2^\nu \Gamma(\nu + 1/2)}, \quad \nu > -1/2.$$

In the limit case $\nu = -1/2$, we put $\pi_{-1/2} = \frac{1}{2}(\delta_{-1} + \delta_1)$. Consequently, for $\alpha \geq -1/2$, the kernel $G_{\alpha,t}(x,y)$ can be expressed as

$$G_{\alpha,t}(x,y) = \big(\sinh(2t)\big)^{-1-\alpha} \int_{-1}^1 \exp\Big(-\frac{1}{2}\coth(2t)(x^2+y^2) - \frac{xys}{\sinh(2t)}\Big)\, \Pi_\alpha(ds).$$

Let

$$q_\pm = q_\pm(x,y,s) = x^2 + y^2 \pm 2xys.$$

Meda's change of variable

$$t = \frac{1}{2}\log\frac{1+\xi}{1-\xi}, \quad \xi \in (0,1), \tag{3.9}$$

leads to

$$G_{\alpha,t}(x,y)=\Big(\frac{1-\xi^2}{2\xi}\Big)^{1+\alpha}\int_{-1}^{1}\exp\Big(-\frac{1}{4\xi}q_+(x,y,s)-\frac{\xi}{4}q_-(x,y,s)\Big)\,\Pi_\alpha(ds).$$

Let

$$\beta_\alpha(\xi)=\sqrt{2\pi}\Big(\frac{1-\xi^2}{2\xi}\Big)^{\alpha+1}\frac{1}{1-\xi^2}\Big(\log\Big(\frac{1+\xi}{1-\xi}\Big)\Big)^{-1/2}.$$

In this way, by (3.9) we get

$$\begin{aligned}R^\alpha(x,y)&=\int_0^1\beta_\alpha(\xi)\delta_\alpha\int_{-1}^{1}\exp\Big(-\frac{q_+}{4\xi}-\frac{\xi q_-}{4}\Big)\,\Pi_\alpha(ds)\,d\xi\\ &=\int_{-1}^{1}\int_0^1\beta_\alpha(\xi)\Big(x-\frac{1}{2\xi}(x+ys)-\frac{\xi}{2}(x-ys)\Big)\exp\Big(-\frac{q_+}{4\xi}-\frac{\xi q_-}{4}\Big)\,d\xi\,\Pi_\alpha(ds).\end{aligned}\tag{3.10}$$

Application of Fubini's theorem above can be justified, see [16, Proposition 5.6].

Throughout the proofs in this section, we will use several elementary facts, that are listed below. First, observe that

$$\beta_\alpha(\xi)\le C2^{-\alpha}\begin{cases}\xi^{-\alpha-3/2}, & 0<\xi\le 1/2,\\ \xi^{-\alpha-1}(1-\xi^2)^\alpha(-\log(1-\xi^2))^{-1/2}, & 1/2<\xi<1,\end{cases}\tag{3.11}$$

$$|x-ys|\le\sqrt{q_-}\tag{3.12}$$

and

$$|x-ys|\exp\Big(-\frac{\xi q_-}{4}\Big)\le C\xi^{-1/2}.\tag{3.13}$$

The inequality (3.12) is immediate, and (3.13) follows from (3.12) and the inequality

$$x^\gamma e^{-x}\le\gamma^\gamma e^{-\gamma},\quad x\in\mathbb{R}_+,\quad \gamma\in\mathbb{R}_+.\tag{3.14}$$

Let $b>0$. Define $h(u):=(1-u)^b u^{v-1/2}$, for $u\in(0,1)$. Then, for $v\ge 1/2$

$$h(u)\le\Big(\frac{b}{b+v-1/2}\Big)^b.\tag{3.15}$$

We will also use frequently the following fact without further mention

$$\frac{\Gamma(z+r)}{\Gamma(z+t)}\simeq z^{r-t},\quad z>0,\quad r,t\in\mathbb{R}.$$

Apart from this, we need several technical lemmas that provide the tools to prove the main estimates. First we show the estimates for the measure of the balls $B(x,(|x-y|))$ in the space $(\mathbb{R}_+,d\mu_\alpha)$. The result presented here is a one-dimensional version of [16, Proposition 3.2]. It will be used tacitly throughout the proofs.

Lemma 3.6. *Let $\alpha\ge -1/2$. Then, for all $x,y\in\mathbb{R}_+$,*

$$\mu_\alpha(B(x,|x-y|))\simeq|x-y|(x+y+|x-y|)^{2\alpha+1}\simeq|x-y|(x+y)^{2\alpha+1}.$$

We will also use the following.

Lemma 3.7. *Let $\xi\in(0,1)$ and $k,m\in\mathbb{R}$ be such that $k+m>-1/2$. Then*

$$\int_0^1\xi^{-k}\beta_m(\xi)\exp\Big(-\frac{q_+}{4\xi}\Big)\,d\xi\le\frac{2^m\Gamma(m+k+1/2)}{q_+^{m+k+1/2}}.$$

Proof. Split the integral into two parts, $\int_0^{1/2}+\int_{1/2}^1$. For the first integral, the result follows from (3.11) and the following estimate

$$\int_0^1 \xi^{-a-1}e^{-T/\xi}\,d\xi \le T^{-a}\Gamma(a), \quad a>0,$$

which, in turn, is a slight modification of [14, Lemma 2.1]. For the second integral, from (3.11), the task is reduced to estimating

$$2^{-m}\int_0^{1/2}\xi^{-m-k-1}(1-\xi^2)^m(-\log(1-\xi^2))^{-1/2}\exp\Big(-\frac{q_+}{4\xi}\Big)\,d\xi.$$

Now, since $\xi\in(1/2,1)$, by (3.14) and Stirling's formula we get

$$\begin{aligned}\xi^{-m-k-1}\exp\big(-\frac{q_+}{4\xi}\big) &\le C\xi\cdot\xi^{-m-k-1/2}\exp\big(-\frac{q_+}{4\xi}\big)\\ &\le C\xi\Big(\frac{4}{q_+}\Big)^{m+k+1/2}(m+k+1/2)^{m+k+1/2}e^{-(m+k+1/2)}\\ &\simeq C\xi\Big(\frac{4}{q_+}\Big)^{m+k+1/2}\Gamma(m+k+3/2)(m+k+1/2)^{-1/2}\\ &= C\xi\Big(\frac{4}{q_+}\Big)^{m+k+1/2}\Gamma(m+k+1/2)(m+k+1/2)^{1/2}.\end{aligned}$$

With this, we get

$$\begin{aligned}2^{-m}\int_0^{1/2}&\xi^{-m-k-1}(1-\xi^2)^m(-\log(1-\xi^2))^{-1/2}\exp\Big(-\frac{q_+}{4\xi}\Big)\,d\xi\\ &\le C\Big(\frac{4}{q_+}\Big)^{m+k+1/2}\Gamma(m+k+1/2)(m+k+1/2)^{1/2}\\ &\quad\times 2^{-m}\int_{1/2}^1\xi(1-\xi^2)^m(-\log(1-\xi^2))^{-1/2}\,d\xi\Big)\\ &\le C\frac{2^m}{q_+^{m+k+1/2}}\,\Gamma(m+k+1/2),\end{aligned}$$

where in the last step we made the change of variable $-\log(1-\xi^2)=w$ in the integral and noticed that it is bounded by $\int_0^\infty e^{-(m+1)w}w^{-1/2}\,dw=\frac{\Gamma(1/2)}{(m+1)^{1/2}}$. □

The lemma below is Lemma 5.3 in [5].

Lemma 3.8. *Let $c\ge -1/2$, $0<B<A$, $\lambda>0$ and $d\ge 0$. Then*

$$\int_0^1\frac{(1-s)^{c+d-1/2}}{(A-Bs)^{c+d+\lambda+1/2}}\,ds\le\frac{C(d)}{A^{c+1/2}B^d(A-B)^\lambda},$$

where

$$C(d)=C_c\begin{cases}\frac{\Gamma(d)\Gamma(\lambda)}{\Gamma(d+\lambda)}, & d>0,\\ 1, & d=0.\end{cases}$$

Now we pass to the proof of Proposition 3.4. Remember that we will prove (3.6) an (3.7) for $j\ge 1$.

3.1.1. *Growth estimates: proof of* (3.6). Let the kernel $R^{\alpha+aj}(x,y)$ be as in (3.10). We write

$$R^{\alpha+aj}(x,y)=J_1-\frac{1}{2}J_2-\frac{1}{2}J_3,$$

where

$$J_1:=x\int_{-1}^{1}\int_0^1\beta_{\alpha+aj}(\xi)\exp\Big(-\frac{q_+}{4\xi}-\frac{\xi q_-}{4}\Big)\,d\xi\,\Pi_{\alpha+aj}(ds),$$

$$J_2:=\int_{-1}^{1}(x+ys)\int_0^1\xi^{-1}\beta_{\alpha+aj}(\xi)\exp\Big(-\frac{q_+}{4\xi}-\frac{\xi q_-}{4}\Big)\,d\xi\,\Pi_{\alpha+aj}(ds),$$

and

$$J_3:=\int_{-1}^{1}(x-ys)\int_0^1\xi\beta_{\alpha+aj}(\xi)\exp\Big(-\frac{q_+}{4\xi}-\frac{\xi q_-}{4}\Big)\,d\xi\,\Pi_{\alpha+aj}(ds).$$

Note that each of the three integrands of the inner integrals above are positive.

Let us begin with the study of J_2. First, note that

$$\begin{aligned}|J_2|&\le|x-y|\int_{-1}^{1}\int_0^1\xi^{-1}\beta_{\alpha+aj}(\xi)\exp\Big(-\frac{q_+}{4\xi}\Big)\,d\xi\,\Pi_{\alpha+aj}(ds)\\&\quad+y\int_{-1}^{1}(1+s)\int_0^1\xi^{-1}\beta_{\alpha+aj}(\xi)\exp\Big(-\frac{q_+}{4\xi}\Big)\,d\xi\,\Pi_{\alpha+aj}(ds):=J_{21}+J_{22}.\end{aligned}$$

By Lemma 3.7 with $k=1$ and $m=\alpha+aj$, the change $s=1-2u$ (that we will use tacitly along this subsection), the inequality $u^{\alpha+aj-1/2}\le 1$, and Lemma 3.8 with $c=\alpha$, $d=aj$ and $\lambda=1$, it is shown that

$$\begin{aligned}J_{21}&\le C|x-y|\frac{\Gamma(\alpha+aj+3/2)}{\Gamma(\alpha+aj+1/2)}\int_{-1}^{1}\frac{(1-s^2)^{\alpha+aj-1/2}}{q_+^{\alpha+aj+3/2}}\,ds\\&\le C|x-y|\frac{\Gamma(\alpha+aj+3/2)4^{\alpha+aj}}{\Gamma(\alpha+aj+1/2)}\int_0^1\frac{u^{\alpha+aj-1/2}(1-u)^{\alpha+aj-1/2}}{((x+y)^2-4xyu)^{\alpha+aj+3/2}}\,du\\&\le C_\alpha|x-y|\frac{\Gamma(\alpha+aj+3/2)4^{\alpha+aj}}{\Gamma(\alpha+aj+1/2)}\frac{\Gamma(aj)}{\Gamma(aj+1)}\frac{1}{(x+y)^{2\alpha+1}}\frac{1}{(4xy)^{aj}}\frac{1}{|x-y|^2}\\&\le\frac{C_\alpha}{(xy)^{aj}\mu_\alpha(B(x,|x-y|))}.\end{aligned}$$

Concerning J_{22}, we consider two cases. First, the case $y>2x$ is immediate, because it can be easily seen that $J_{22}\le J_{21}$. For the other case, when $y\le 2x$, Lemma 3.7 and (3.15) with $v=\alpha+aj$ and $b=1/2$ lead to

$$\begin{aligned}J_{22}&\le C\frac{\sqrt{xy}\,\Gamma(\alpha+aj+3/2)}{\Gamma(\alpha+aj+1/2)}\int_{-1}^{1}\frac{(1+s)(1-s^2)^{\alpha+aj-1/2}}{q_+^{\alpha+aj+3/2}}\,ds\\&=C\frac{\sqrt{xy}\,\Gamma(\alpha+aj+3/2)4^{\alpha+aj}}{\Gamma(\alpha+aj+1/2)}\int_0^1\frac{u^{\alpha+aj-1/2}(1-u)^{\alpha+aj+1/2}}{((x+y)^2-4xyu)^{\alpha+aj+3/2}}\,du\\&\le C\frac{\sqrt{xy}\,\Gamma(\alpha+aj+3/2)4^{\alpha+aj}}{\Gamma(\alpha+aj+1/2)\sqrt{\alpha+aj}}\int_0^1\frac{(1-u)^{\alpha+aj}}{((x+y)^2-4xyu)^{\alpha+aj+3/2}}\,du\\&\le C_\alpha\frac{\sqrt{xy}\,\Gamma(\alpha+aj+3/2)4^{\alpha+aj}}{\Gamma(\alpha+aj+1/2)\sqrt{\alpha+aj}}\frac{\Gamma(aj+1/2)}{\Gamma(aj+1)}\frac{1}{(x+y)^{2\alpha+1}}\frac{1}{(4xy)^{aj+1/2}}\frac{1}{|x-y|}\\&\le\frac{C_\alpha}{(xy)^{aj}\mu_\alpha(B(x,|x-y|))}\end{aligned}$$

where, in the last step, we applied Lemma 3.8 with $c = \alpha$, $d = aj+1/2$ and $\lambda = 1/2$.

We continue with J_3. It follows from (3.13) and Lemma 3.7 with $k = 1/2$ and $m = \alpha + aj$ that

$$\begin{aligned}|J_3| &\le C\int_{-1}^{1}\int_0^1 \xi^{1/2}\beta_{\alpha+aj}(\xi)\exp\Big(-\frac{q_+}{4\xi}\Big)\,d\xi\,\Pi_{\alpha+aj}(ds)\\ &\le \int_{-1}^{1}\int_0^1 \xi^{-1/2}\beta_{\alpha+aj}(\xi)\exp\Big(-\frac{q_+}{4\xi}\Big)\,d\xi\,\Pi_{\alpha+aj}(ds)\\ &\le C2^{\alpha+aj}\Gamma(\alpha+aj+1)\int_{-1}^{1}\frac{\Pi_{\alpha+aj}(ds)}{q_+^{\alpha+aj+1}}.\end{aligned}$$

Thus, we arrive at

$$\begin{aligned}|J_3| &\le C\frac{\Gamma(\alpha+aj+1)}{\Gamma(\alpha+aj+1/2)}\int_{-1}^{1}\frac{(1-s^2)^{\alpha+aj-1/2}}{q_+^{\alpha+aj+1}}\,ds\\ &= C\frac{4^{\alpha+aj}\Gamma(\alpha+aj+1)}{\Gamma(\alpha+aj+1/2)}\int_0^1\frac{u^{\alpha+aj-1/2}(1-u)^{\alpha+aj-1/2}}{((x+y)^2-4xyu)^{\alpha+aj+1}}\,du\\ &\le C_\alpha\frac{4^{\alpha}\Gamma(\alpha+aj+1)}{\Gamma(\alpha+aj+1/2)}\frac{\Gamma(aj)}{\Gamma(aj+1/2)}\frac{1}{(x+y)^{2\alpha+1}}\frac{1}{(xy)^{aj}}\frac{1}{|x-y|}\\ &\le \frac{C_\alpha}{(xy)^{aj}\mu_\alpha(B(x,|x-y|))},\end{aligned}$$

after using that $u^{\alpha+aj-1/2}\le 1$ and Lemma 3.8 with $c=\alpha$, $d = aj$ and $\lambda = 1/2$.

Finally, we study J_1. We split the outer integral into two parts, $J_1 = x\int_{-1}^{0} + x\int_0^1 := xJ_{11} + xJ_{12}$. For the first case, observe that $x < x - sy$. This and (3.13) imply

$$\begin{aligned}x|J_{11}| &\le \int_{-1}^{0}|x-ys|\int_0^1\beta_{\alpha+aj}(\xi)\exp\Big(-\frac{q_+}{4\xi}-\frac{\xi q_-}{4}\Big)\,d\xi\,\Pi_{\alpha+aj}(ds)\\ &\le C\int_{-1}^{1}\int_0^1 \xi^{-1/2}\beta_{\alpha+aj}(\xi)\exp\Big(-\frac{q_+}{4\xi}\Big)\,d\xi\,\Pi_{\alpha+aj}(ds)\end{aligned}$$

and we just proceed as in the estimate for J_3. Concerning xJ_{12}, we have $x < x+ys$ in this case. Then

$$x|J_{12}| \le \int_{-1}^{1}|x+ys|\int_0^1\xi^{-1}\beta_{\alpha+aj}(\xi)\exp\Big(-\frac{q_+}{4\xi}\Big)\,d\xi\,\Pi_{\alpha+aj}(ds),$$

so this case is reduced to that one of J_2, and we are done.

3.1.2. *Smoothness estimates: proof of* (3.7). Observe that

$$\frac{d}{dx}[(xy)^{aj}R^{\alpha+aj}(x,y)] = (aj)x^{aj-1}y^{aj}R^{\alpha+aj}(x,y) + (xy)^{aj}\frac{d}{dx}(R^{\alpha+aj}(x,y)). \tag{3.16}$$

Therefore, our first aim is to get the estimate

$$\frac{aj}{x}R^{\alpha+aj}(x,y) \le \frac{C}{(xy)^{aj}|x-y|\mu_\alpha(B(x,|x-y|))}.$$

Recall from the previous subsection that $R^{\alpha+aj}(x,y)$ can be written in terms of three expressions J_1, J_2 and J_3. We will prove the estimate above for each one of the corresponding expressions.

Let us begin with J_2. With the same reasoning as in the previous subsubsection, observe that

$$\frac{aj}{x}|J_2| \leq C\frac{aj}{x}\frac{\Gamma(\alpha+aj+3/2)4^{\alpha+aj}}{\Gamma(\alpha+aj+1/2)}\Big(|x-y|\int_0^1 \frac{u^{\alpha+aj-1/2}(1-u)^{\alpha+aj-1/2}}{((x+y)^2-4xyu)^{\alpha+aj+3/2}}\,du + y\int_0^1 \frac{u^{\alpha+aj-1/2}(1-u)^{\alpha+aj+1/2}}{((x+y)^2-4xyu)^{\alpha+aj+3/2}}\,du\Big) := I_{21}+I_{22}.$$

Consider now two cases. First, if $2x \geq y$. For I_{21}, by (3.15) with $v=\alpha+aj$ and $b=1/2$ and Lemma 3.8 with $c=\alpha$, $d=aj-1/2$ and $\lambda=3/2$, we have

$$\begin{aligned} I_{21} &\leq C\frac{|x-y|}{x}\frac{\Gamma(\alpha+aj+3/2)4^{\alpha+aj}}{\Gamma(\alpha+aj+1/2)}\frac{aj}{\sqrt{\alpha+aj}}\int_0^1 \frac{(1-u)^{\alpha+aj-1}}{((x+y)^2-4xyu)^{\alpha+aj+3/2}}\,du \\ &\leq C_\alpha\frac{|x-y|}{x}\frac{\Gamma(\alpha+aj+3/2)4^{\alpha+aj}}{\Gamma(\alpha+aj+1/2)}\frac{aj}{\sqrt{\alpha+aj}}\frac{\Gamma(aj-1/2)}{\Gamma(aj+1)} \\ &\qquad\times\frac{1}{(x+y)^{2\alpha+1}}\frac{1}{(4xy)^{aj-1/2}}\frac{1}{|x-y|^3} \\ &\leq \frac{C_\alpha}{(xy)^{aj}|x-y|\mu_\alpha(B(x,|x-y|))}. \end{aligned}$$

Concerning I_{22}, by (3.15) with $v=\alpha+aj$ and $b=1$ and Lemma 3.8 with $c=\alpha$, $d=aj$ and $\lambda=1$

$$\begin{aligned} I_{22} &\leq C\frac{\Gamma(\alpha+aj+3/2)4^{\alpha+aj}}{\Gamma(\alpha+aj+1/2)}\frac{aj}{\alpha+aj+1/2}\int_0^1 \frac{(1-u)^{\alpha+aj-1/2}}{((x+y)^2-4xyu)^{\alpha+aj+3/2}}\,du \\ &\leq C_\alpha\frac{\Gamma(\alpha+aj+3/2)4^{\alpha+aj}}{\Gamma(\alpha+aj+1/2)}\frac{aj}{\alpha+aj+1/2}\frac{\Gamma(aj)}{\Gamma(aj+1)} \\ &\qquad\times\frac{1}{(x+y)^{2\alpha+1}}\frac{1}{(4xy)^{aj}}\frac{1}{|x-y|^2} \\ &\leq \frac{C_\alpha}{(xy)^{aj}|x-y|\mu_\alpha(B(x,|x-y|))}. \end{aligned}$$

Secondly, if $2x<y$. For I_{21}, observe that $|x-y|\sim y$, use that $u^{\alpha+aj-1/2}(1-u)\leq C$ and apply Lemma 3.8 with $c=\alpha$, $d=aj-1$ and $\lambda=2$, thus

$$\begin{aligned} I_{21} &\leq Caj\frac{|x-y|}{x}\frac{\Gamma(\alpha+aj+3/2)4^{\alpha+aj}}{\Gamma(\alpha+aj+1/2)}\int_0^1 \frac{(1-u)^{\alpha+aj-3/2}}{((x+y)^2-4xyu)^{\alpha+aj+3/2}}\,du \\ &\leq Caj\frac{|x-y|^2}{xy}\frac{\Gamma(\alpha+aj+3/2)4^{\alpha+aj}}{\Gamma(\alpha+aj+1/2)}\frac{\Gamma(aj-1)}{\Gamma(aj+1)}\frac{1}{(x+y)^{2\alpha+1}}\frac{1}{(4xy)^{aj-1}}\frac{1}{|x-y|^4} \\ &\leq \frac{C_\alpha}{(xy)^{aj}|x-y|\mu_\alpha(B(x,|x-y|))}. \end{aligned}$$

For I_{22}, using that $u^{\alpha+aj-1/2}(1-u)^2\leq C$, it is clear that

$$I_{22} \leq Caj\frac{|x-y|}{x}\frac{\Gamma(\alpha+aj+3/2)4^{\alpha+aj}}{\Gamma(\alpha+aj+1/2)}\int_0^1 \frac{(1-u)^{\alpha+aj-3/2}}{((x+y)^2-4xyu)^{\alpha+aj+3/2}}\,du$$

and this case follows analogously to the study of I_{21} for $2x<y$.

Now we pass to $\frac{aj}{x}J_3$. Reasoning as in the previous subsubsection for J_3, we have

$$\frac{aj}{x}|J_3| \leq C\frac{aj}{x}\frac{4^{\alpha+aj}\Gamma(\alpha+aj+1)}{\Gamma(\alpha+aj+1/2)}\int_0^1 \frac{u^{\alpha+aj-1/2}(1-u)^{\alpha+aj-1/2}}{((x+y)^2-4xyu)^{\alpha+aj+1}}\,du.$$

We distinguish two cases. If $2x \ge y$, we use (3.15) with $v = \alpha + aj$ and $b = 1/2$ and Lemma 3.8 with $c = \alpha$, $d = aj - 1/2$ and $\lambda = 1$ in order to get the result. If $2x < y$, the fact that $u^{\alpha+aj-1/2}(1-u) \le C$ and Lemma 3.8 with $c = \alpha$, $d = aj - 1$ and $\lambda = 3/2$ yield the desired estimate.

Finally, with regard to $\frac{aj}{x}J_1$, we split the outer integral into two parts, $\frac{aj}{x}J_1 = \frac{aj}{x}x\int_{-1}^0 + \frac{aj}{x}x\int_0^1 := \frac{aj}{x}xI_{11} + \frac{aj}{x}xI_{12}$. As for the first summand, analogously to the treatment of xJ_{11} in the previous subsubsection, by (3.13) we have

$$\frac{aj}{x}x|I_{11}| \le C\frac{aj}{x}\int_{-1}^1\int_0^1 \xi^{-1/2}\beta_{\alpha+aj}(\xi)\exp\Big(-\frac{q_+}{4\xi}\Big)\,d\xi\,\Pi_{\alpha+aj}(ds),$$

and this case is reduced to the study of $\frac{aj}{x}|J_3|$ above. Concerning I_{12}, observe that, with analogous reasonings as in the previous subsubsection,

$$\frac{aj}{x}x\,|I_{12}| \le C\frac{aj}{x}\int_{-1}^1 |x+ys|\int_0^1 \xi^{-1}\beta_{\alpha+aj}(\xi)\exp\Big(-\frac{q_+}{4\xi}\Big)\,d\xi\,\Pi_{\alpha+aj}(ds).$$

and we treat the last expression as we did with $\frac{aj}{x}|J_2|$.

Taking into account (3.16), we proceed now with $(xy)^{aj}\frac{d}{dx}(R^{\alpha+aj}(x,y))$. From the expression for the kernel in (3.10), we get

$$\begin{aligned}
&(xy)^{aj}\frac{d}{dx}(R^{\alpha+aj}(x,y))\\
&= (xy)^{aj}\int_{-1}^1\int_0^1\Big(1-\frac{1}{2\xi}-\frac{\xi}{2}\Big)\beta_{\alpha+aj}(\xi)\exp\Big(-\frac{q_+}{4\xi}-\frac{\xi q_-}{4}\Big)\,d\xi\,\Pi_{\alpha+aj}(ds)\\
&\quad + (xy)^{aj}\int_{-1}^1\int_0^1\Big(x-\frac{(x+ys)}{2\xi}-\frac{(x-ys)\xi}{2}\Big)\\
&\quad \times\Big(-\Big(\frac{x+ys}{2}\Big)\frac{1}{\xi}-\Big(\frac{x-ys}{2}\Big)\xi\Big)\beta_{\alpha+aj}(\xi)\exp\Big(-\frac{q_+}{4\xi}-\frac{\xi q_-}{4}\Big)\,d\xi\,\Pi_{\alpha+aj}(ds)\\
&:= (xy)^{aj}(S_1+S_2).
\end{aligned}$$

Concerning S_1, by Lemma 3.7 with $k = 1$ and $m = \alpha + aj$ and Lemma 3.8 with $c = \alpha$, $d = aj$ and $\lambda = 1$ we get

$$\begin{aligned}
S_1 &\le C\int_{-1}^1\int_0^1 \xi^{-1}\beta_{\alpha+aj}(\xi)\exp\Big(-\frac{q_+}{4\xi}\Big)\,d\xi\,\Pi_{\alpha+aj}(ds)\\
&\le C\frac{\Gamma(\alpha+aj+3/2)}{\Gamma(\alpha+aj+1/2)}\int_{-1}^1\frac{(1-s^2)^{\alpha+aj-1/2}}{q_+^{\alpha+aj+3/2}}\,ds\\
&= C_\alpha\frac{\Gamma(\alpha+aj+3/2)}{\Gamma(\alpha+aj+1/2)}\frac{\Gamma(aj)}{\Gamma(aj+1)}\frac{4^{\alpha+aj}}{(x+y)^{2\alpha+1}}\frac{1}{(4xy)^{aj}}\frac{1}{|x-y|^2}\\
&\le \frac{C_\alpha}{(xy)^{aj}(x+y)^{2\alpha+1}|x-y|^2} \simeq \frac{C_\alpha}{(xy)^{aj}|x-y|\mu_\alpha(B(x,|x-y|))}.
\end{aligned}$$

The study of S_2 is more involved. We write

$$\begin{aligned}
S_2 &= \sum_{j=1}^5 S_{2j}\\
&\qquad = \sum_{j=1}^5\int_{-1}^1\int_0^1 z_j(x,y,\xi)\beta_{\alpha+aj}(\xi)\exp\Big(-\frac{q_+}{4\xi}-\frac{\xi q_-}{4}\Big)\,d\xi\,\Pi_{\alpha+aj}(ds),
\end{aligned}$$

where $z_1(x,y,\xi) = -x\Big(\frac{x+ys}{2}\Big)\frac{1}{\xi}$; $z_2(x,y,\xi) = -x\Big(\frac{x-ys}{2}\Big)\xi$; $z_3(x,y,\xi) = \Big(\frac{x+ys}{2\xi}\Big)^2$; $z_4(x,y,\xi) = \frac{(x^2-y^2s^2)}{2}$; and $z_5(x,y,\xi) = \Big(\frac{(x-ys)\xi}{2}\Big)^2$. Observe that, if $s<0$ then $|z_1| \le \frac{|x^2-y^2s^2|}{2\xi}$; otherwise, if $s>0$ then $|z_1| \le \Big(\frac{x+ys}{\xi}\Big)^2 = |z_3|$. Note also that $z_4 \le C\frac{x^2-y^2s^2}{2\xi}$. Therefore, $|S_{21}| \le |S_{23}| + Q$, where

$$Q := \int_{-1}^1\int_0^1 \frac{|x^2-y^2s^2|}{\xi}\beta_{\alpha+aj}(\xi)\exp\Big(-\frac{q_+}{4\xi}-\frac{\xi q_-}{4}\Big)\,d\xi\,\Pi_{\alpha+aj}(ds).$$

In this way, if we get the desired estimates for $|S_{23}|$ and Q, then we immediately get the same estimates for $|S_{21}|$ and $|S_{24}|$. Concerning Q, by (3.13) and Lemma 3.7 with $k=3/2$ and $m=\alpha+aj$, we get

$$\begin{aligned}
Q &\le C\int_{-1}^1\int_0^1 \frac{(x+ys)}{\xi^{3/2}}\beta_{\alpha+aj}(\xi)\exp\Big(-\frac{q_+}{4\xi}\Big)\,d\xi\,\Pi_{\alpha+aj}(ds)\\
&\le C|x-y|\frac{\Gamma(\alpha+aj+2)4^{\alpha+aj}}{\Gamma(\alpha+aj+1/2)}\int_0^1 \frac{(1-u)^{\alpha+aj-1/2}}{((x+y)^2-4xyu)^{\alpha+aj+2}}\,du\\
&\quad + Cy\frac{\Gamma(\alpha+aj+2)4^{\alpha+aj}}{\Gamma(\alpha+aj+1/2)}\int_0^1 \frac{u^{\alpha+aj-1/2}(1-u)^{\alpha+aj+1/2}}{((x+y)^2-4xyu)^{\alpha+aj+2}}\,du := Q_1+Q_2.
\end{aligned}$$

The estimate for Q_1 follows immediately from Lemma 3.8 with $c=\alpha$, $d=aj$ and $\lambda=3/2$. As for Q_2, we use (3.15) with $v=\alpha+aj$ and $b=1/2$ and Lemma 3.8 with $c=\alpha+1/2$, $d=aj$ and $\lambda=1$ to obtain

$$\begin{aligned}
Q_2 &\le C_\alpha(x+y)\frac{\Gamma(\alpha+aj+2)4^{\alpha+aj}}{\Gamma(\alpha+aj+1/2)\sqrt{\alpha+aj}}\frac{\Gamma(aj)}{\Gamma(aj+1)}\frac{1}{(x+y)^{2\alpha+2}}\frac{1}{(4xy)^{aj}}\frac{1}{|x-y|^2}\\
&\le \frac{C_\alpha}{(xy)^{aj}|x-y|\mu_\alpha(B(x,|x-y|))}.
\end{aligned}$$

Now, for $|S_{23}|$ we can write

$$\begin{aligned}
|S_{23}| &\le C|x-y|^2\int_{-1}^1\int_0^1 \xi^{-2}\beta_{\alpha+aj}(\xi)\exp\Big(-\frac{q_+}{4\xi}-\frac{\xi q_-}{4}\Big)\Big)\,d\xi\,\Pi_{\alpha+aj}(ds)\\
&\quad + Cy^2\int_{-1}^1(1+s)^2\int_0^1 \xi^{-2}\beta_{\alpha+aj}(\xi)\exp\Big(-\frac{q_+}{4\xi}-\frac{\xi q_-}{4}\Big)\Big)\,d\xi\,\Pi_{\alpha+aj}(ds)\\
&\quad := T_1+T_2.
\end{aligned}$$

Observe that, by Lemma 3.7 with $k=2$ and $m=\alpha+aj$ and Lemma 3.8 with $c=\alpha$, $d=aj$ and $\lambda=2$, the estimate for T_1 follows immediately. Concerning T_2 we distinguish two cases. If $y>2x$, then $y^2\simeq|x-y|^2$ and this case is reduced to the study of T_1. If $y\le 2x$, then by Lemma 3.7 with $k=2$ and $m=\alpha+aj$,

$$\begin{aligned}
T_2 &\le Cxy\frac{\Gamma(\alpha+aj+5/2)4^{\alpha+aj}}{\Gamma(\alpha+aj+1/2)}\int_0^1 \frac{u^{\alpha+aj-1/2}(1-u)^{\alpha+aj+3/2}}{((x+y)^2-4xyu)^{\alpha+aj+5/2}}\,du\\
&\le C_\alpha xy\frac{\Gamma(\alpha+aj+5/2)4^{\alpha+aj}}{\Gamma(\alpha+aj+1/2)(\alpha+aj)}\int_0^1 \frac{(1-u)^{\alpha+aj+1/2}}{((x+y)^2-4xyu)^{\alpha+aj+5/2}}\,du\\
&\le C_a\frac{\Gamma(\alpha+aj+5/2)4^{\alpha+aj}}{(\alpha+aj)\Gamma(\alpha+aj+1/2)}\frac{\Gamma(aj+1)}{\Gamma(aj+2)}\frac{1}{(x+y)^{2\alpha+1}}\frac{1}{(4xy)^{aj}}\frac{1}{|x-y|^2}
\end{aligned}$$

$$\leq \frac{C_\alpha}{(xy)^{aj}|x-y|\mu_\alpha(B(x,|x-y|))},$$

where we used (3.15) with $v=\alpha+aj$ and $b=1$ and Lemma 3.8 with $c=\alpha$, $d=aj+1$ and $\lambda=1$.

For S_{22}, see that

$$\begin{aligned}|S_{22}| &\leq C\int_{-1}^{1}\int_0^1 |x+ys|\frac{|x-ys|}{2}\xi\beta_{\alpha+aj}(\xi)\exp\Big(-\frac{q_+}{4\xi}-\frac{\xi q_-}{4}\Big)\,d\xi\,\Pi_{\alpha+aj}(ds)\\ &\quad + C\int_{-1}^{1}\int_0^1 |x-ys|\frac{|x-ys|}{2}\xi\beta_{\alpha+aj}(\xi)\exp\Big(-\frac{q_+}{4\xi}-\frac{\xi q_-}{4}\Big)\,d\xi\,\Pi_{\alpha+aj}(ds)\\ &\leq S_{24}+P,\end{aligned}$$

where P is the second integral. The factor S_{24} was already studied above, since that case was reduced to the study of Q. On the other hand, we use (3.13), and by Lemma 3.7 with $k=1$ and $m=\alpha+aj$ and Lemma 3.8 with $c=\alpha$, $d=aj$ and $\lambda=1$, we get

$$\begin{aligned}P &\leq C\int_{-1}^{1}\int_0^1 \xi^{-1}\beta_{\alpha+aj}(\xi)\exp\Big(-\frac{q_+}{4\xi}\Big)\,d\xi\,\Pi_{\alpha+aj}(ds)\\ &\leq C\frac{\Gamma(\alpha+aj+3/2)4^{\alpha+aj}}{\Gamma(\alpha+aj+1/2)}\frac{\Gamma(aj)}{\Gamma(aj+1)}\frac{1}{(x+y)^{2\alpha+1}}\frac{1}{(4xy)^{aj}}\frac{1}{|x-y|^2}\\ &\leq \frac{C_\alpha}{(xy)^{aj}|x-y|\mu_\alpha(B(x,|x-y|))}.\end{aligned}$$

Finally, note that $S_{25}\leq P$ and we are done.

3.2. **Proof of Proposition 3.3.** The proof of Proposition 3.3 is an immediate consequence of the extrapolation theorem of Rubio de Francia, see [18, 19], adapted to our setting. Such adaptation is the following.

Theorem 3.9. *Suppose that for some pair of nonnegative functions (f,g), for some fixed $1\leq p<\infty$ and for all $w\in A_p^\alpha$ we have*

$$\int_0^\infty g(x)^p w(x)\,d\mu_\alpha(x)\leq C\int_0^\infty f(x)^p w(x)\,d\mu_\alpha(x),$$

with C depending only on w. Then, for all $1<r<\infty$ and all $w\in A_r^\alpha$ we have

$$\int_0^\infty g(x)^r w(x)\,d\mu_\alpha(x)\leq C\int_0^\infty f(x)^r w(x)\,d\mu_\alpha(x).$$

Originally the extrapolation theorem was given for sublinear operators, but it turns out that sublinearity is not necessary. Actually even the operator itself does not play any role and all the statements can be given in terms of pairs of nonnegative measurable functions. This observation was made by D. Cruz-Uribe and C. Pérez in [8, Remark 1.11] and then J. Duoandikoetxea adopted this setting in [9]. We also follow this idea here.

The proof of this kind of results uses Rubio de Francia's algorithm (see [19] and [9, Lemma 2.2]). In order to apply the algorithm in our setting we need an operator bounded in $L^p(\mathbb{R}_+,w\,d\mu_\alpha)$ for weights $w\in A_p^\alpha$, with $p>1$. The required operator is a maximal operator of Hardy-Littlewood type,

$$M_\alpha f(x)=\sup_{x\in I}\frac{1}{\mu_\alpha(I)}\int_I |f(y)|\,d\mu_\alpha(y),$$

of which mapping properties with weights in the A_p^α class follow from a more general result proved in [3] by A. P. Calderón.

To deduce Proposition 3.3 from Theorem 3.9 we proceed in the following way. With Theorem 3.9 we extend (3.4) to any r, with $1 < r < \infty$. Now, (3.5) with $p = r$ is immediate. Finally, applying Theorem 3.9 with

$$g = \Big(\sum_{j=0}^{\infty} |T_j f_j|^r\Big)^{1/r} \qquad \text{and} \qquad f = \Big(\sum_{j=0}^{\infty} |f_j|^r\Big)^{1/r}$$

the proof is completed.

4. Vector-valued inequalities for the fractional integrals for the Laguerre expansions of convolution type

This section is devoted to the analysis of a vector-valued inequality for the fractional integrals of the Laguerre expansions of convolution type. It arises associated to the angular part, $\frac{1}{r}\nabla_0$, of the operator δ defined in (2.2) and in the definition of the Riesz transform for the harmonic oscillator.

Let $(L_\alpha)^{-1/2}$ be the fractional integral of order $1/2$ for the Laguerre expansions as was given in (3.2). Then, we define the operator

$$\mathcal{T}^\alpha f(x) = \frac{1}{x}(L_\alpha)^{-1/2} f(x), \quad x \in \mathbb{R}_+.$$

The boundedness properties of this operator are contained in the following theorem.

Theorem 4.1. *Let $\alpha > -1$, $a \geq 1$, and $1 < p, r < \infty$. Define $u_j(x) = x^{aj}$, $j = 1, 2, \dots$. Then there exists a constant C such that*

$$\Big\|\Big(\sum_{j=1}^{\infty} |j u_j \mathcal{T}^{\alpha+aj}(u_j^{-1} f_j)|^r\Big)^{1/r}\Big\|_{L^p(\mathbb{R}_+, w\, d\mu_\alpha)} \leq C \Big\|\Big(\sum_{j=0}^{\infty} |f_j|^r\Big)^{1/r}\Big\|_{L^p(\mathbb{R}_+, w\, d\mu_\alpha)}$$

for all $w \in A_p^\alpha$. Moreover the constant C depends only on α and w.

We will deduce the result by using Proposition 3.3 and the following proposition

Proposition 4.2. *Let $\alpha > -1$, $a \geq 1$, and $1 < p < \infty$. Define $u_j(x) = x^{aj}$, $j = 1, 2, \dots$. Then*

$$|j u_j \mathcal{T}^{\alpha+aj}(u_j^{-1} f)(x)|^p w(x)\, d\mu_\alpha(x) \leq C \int_0^\infty |f(x)|^p w(x)\, d\mu_\alpha(x),$$

for every weight $w \in A_p^\alpha$, where C is a constant independent of j and depending on α and w.

In the proof of the previous proposition we will use the Calderón-Zygmund theory and in order to do this we need the L^2 estimate for the operator without weights.

Proposition 4.3. *Let $\alpha > -1$ and $a \geq 1$. Define $u_j(x) = x^{aj}$, $j = 1, 2, \dots$. Then*

$$\int_0^\infty |j u_j \mathcal{T}^{\alpha+aj}(u_j^{-1} f)(x)|^2\, d\mu_\alpha(x) \leq C \int_0^\infty |f(x)|^2\, d\mu_\alpha(x), \tag{4.1}$$

for every $f \in L^2(\mathbb{R}_+, d\mu_\alpha)$, with C independent of j.

Proof. Take $g = u_j^{-1} f$. Then, the inequality (4.1) is equivalent to

$$\int_0^\infty |(\alpha + aj)\mathcal{T}^{\alpha+aj}(g)(x)|^2 \, d\mu_{\alpha+aj}(x) \le C \int_0^\infty |g(x)|^2 \, d\mu_{\alpha+aj}(x). \tag{4.2}$$

From the identities for the Laguerre polynomials [1, p. 783, 22.7.29 and 22.7.30]

$$L_n^{\alpha+1}(x) = \frac{1}{x}((x-n)L_n^\alpha(x) + (\alpha+n)L_{n-1}^\alpha(x))$$

and

$$L_n^{\alpha-1}(x) = L_n^\alpha(x) - L_{n-1}^\alpha(x),$$

we can deduce that

$$\frac{\alpha}{x} L_n^\alpha(x^2) = \frac{n+1}{x} L_{n+1}^{\alpha-1}(x^2) + 2x L_{n+1}^{\alpha+1}(x^2) - x L_n^{\alpha+1}(x^2).$$

and

$$\frac{\alpha}{x}\ell_n^\alpha(x) = \frac{\sqrt{n+1}}{x}\ell_{n+1}^{\alpha-1}(x) + 2x\sqrt{\frac{(n+\alpha+2)(n+\alpha+1)}{n+1}}\ell_{n+1}^{\alpha+1}(x) \\ - x\sqrt{n+\alpha+1}\,\ell_n^{\alpha+1}(x).$$

In this way, the left side of (4.2) is bounded by the sum of

$$\int_0^\infty \Big| \sum_{k=0}^\infty \sqrt{\frac{k+1}{4k+2\alpha+2aj+2}} \frac{\ell_{k+1}^{\alpha+aj-1}(x)}{x} \langle f, \ell_k^{\alpha+aj}\rangle_{d\mu_{\alpha+aj}} \Big|^2 \, d\mu_{\alpha+aj}(x),$$

$$\int_0^\infty \Big| \sum_{k=0}^\infty \sqrt{\frac{(k+\alpha+aj+2)(k+\alpha+aj+1)}{(4k+2\alpha+2aj+2)(k+1)}} x\ell_{k+1}^{\alpha+aj+1}(x)\langle f, \ell_k^{\alpha+aj}\rangle_{d\mu_{\alpha+aj}} \Big|^2 \, d\mu_{\alpha+aj}(x),$$

and

$$\int_0^\infty \Big| \sum_{k=0}^\infty \sqrt{\frac{k+\alpha+aj+1}{4k+2\alpha+2aj+2}} x\ell_k^{\alpha+aj+1}(x)\langle f, \ell_k^{\alpha+aj}\rangle_{d\mu_{\alpha+aj}} \Big|^2 \, d\mu_{\alpha+aj}(x).$$

But the three previous summands can be controlled by the right side of (4.2) with a constant C independent of j. □

The operator $\mathcal{T}^\alpha f$ can be written as

$$\mathcal{T}^\alpha f(x) = \int_0^\infty T^\alpha(x,y) f(y) \, d\mu_\alpha(y)$$

where

$$T^\alpha(x,y) = \frac{1}{x}\frac{1}{\sqrt{\pi}} \int_0^\infty G_{\alpha,t}(x,y) t^{-1/2} \, dt.$$

So, proceeding as in Proposition 3.5, we can see that the operators $ju_j\mathcal{T}^{\alpha+aj}(u_j^{-1}f)$ can be associated, in the Calderón-Zygmund sense, with the kernel

$$j(xy)^{aj} T^{\alpha+aj}(x,y) = \frac{j}{x}\frac{(xy)^{aj}}{\sqrt{\pi}} \int_0^\infty G_{\alpha,t}(x,y) t^{-1/2} \, dt.$$

For this kernel, the following estimates of Calderón-Zygmund are verified.

Proposition 4.4. *Let $\alpha > -1$, $a \geq 1$, and $j \geq 1$. Then*

$$|j(xy)^{aj}T^{\alpha+aj}(x,y)| \leq \frac{C_1}{\mu_\alpha(B(x,|x-y|))}, \quad x \neq y,$$

$$|j\nabla_{x,y}(xy)^{aj}T^{\alpha+aj}(x,y)| \leq \frac{C_2}{|x-y|\mu_\alpha(B(x,|x-y|))}, \quad x \neq y,$$

with C_1 and C_2 independent of j, and where $\mu_\alpha(B(x,|x-y|)) = \int_{B(x,|x-y|)} d\mu_\alpha$ and $B(x,|x-y|)$ is the ball of center x and radius $|x-y|$.

The proof follows the lines of Proposition 3.4 so the details are omitted.

5. Proof of the Theorem 2.1

Proof. With the change $i = m - 2j$, we have

$$|Rf|^2 = |\delta H^{-1/2} f|^2 = \Big|\sum_{i=0}^{\infty} \sum_{\ell=1}^{\dim \mathcal{H}_i} \sum_{j=0}^{\infty} \frac{c_{2j+i,j,\ell}(f)}{\sqrt{n+2i+4j}} \delta\tilde{\phi}_{2j+i,j,\ell}\Big|^2.$$

Now, from the identities (see [17, Lemma 2.2]) $\langle \xi, \nabla_0 \mathcal{Y}_{j,\ell}(x)\rangle = 0$ and

$$\int_{\mathbb{S}^{n-1}} \langle \nabla_0 \mathcal{Y}_{j,\ell}(\xi), \nabla_0 \mathcal{Y}_{j',\ell'}(\xi)\rangle \, d\sigma(\xi) = j(2j+n-2)\delta_{j,j'}\delta_{\ell,\ell'},$$

it can be easily checked that

$$\int_{\mathbb{S}^{n-1}} |Rf(x)|^2 \, d\sigma(\xi) = O_1(f) + O_2(f),$$

where

$$O_1(f) = \sum_{i=0}^{\infty} \sum_{\ell=1}^{\dim \mathcal{H}_i} \left| r^i \mathcal{R}^{n/2-1+i}(r^{-i} f_{i,\ell})\right|^2,$$

and

$$O_2(f) = \sum_{i=1}^{\infty} \sum_{\ell=1}^{\dim \mathcal{H}_i} \frac{2i+n-2}{i} \left| i r^i \mathcal{T}^{n/2-1+i}(r^{-i} f_{i,\ell})\right|^2.$$

Then the proof is an immediate consequence of Theorem 3.1 and Theorem 4.1. □

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